

On semiclassical four-point correlators in $AdS_5 \times S^5$

D. Arnaudov^{*} and R. C. Rashkov^{†, **}

^{*} Department of Physics, Sofia University,
5 J. Bourchier Blvd, 1164 Sofia, Bulgaria

[†] Institute for Theoretical Physics,
Vienna University of Technology,
Wiedner Hauptstr. 8-10, 1040 Vienna, Austria

Abstract

Following the recent advances in the holographic calculation of n -point correlation functions with two “heavy” (with large quantum numbers) states at strong coupling, we extend these findings by computing subleading four-point functions of two heavy and two “light” (supergravity) operators in $\mathcal{N} = 4$ SYM. We also investigate specific correlators of four heavy BMN states.

1 Introduction

The attempts to establish a correspondence between the large N limit of gauge theories and string theory have a long history. Relatively recently an explicit realization of such a duality was provided by the Maldacena conjecture about AdS/CFT correspondence [1]. The great number of convincing results from the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory [1, 2, 3] made this subject a major research area, and many fascinating new features have been established.

One of the results of this duality is the relation between planar correlation functions of single-trace conformal primary operators in the boundary gauge theory and correlators of corresponding closed-string vertex operators on a worldsheet with the topology of S^2 . Let us start with the study of a correlation function with two heavy vertex operators (with large quantum numbers of the order of the string tension) and a number of light vertex operators (with quantum numbers and dimensions of order one). Then

^{*}e-mail: rash@hep.itp.tuwien.ac.at.

the large $\sqrt{\lambda}$ behavior of correlators of such operators is determined by a semiclassical string trajectory governed by the heavy operator insertions, and with sources provided by the vertex operators of the light states. The exact procedure is the following. First we find the classical string solution that determines the leading large $\sqrt{\lambda}$ contribution to the two-point function of the heavy operators. Second we calculate the full correlator by evaluating the product of light vertex operators on this solution.

This semiclassical approach was developed for the calculation of two-point functions in [4]–[8]. An extension of this method to certain three-point correlators was discussed in [9, 8], and elaborated in [10, 11], where the heavy operators corresponded to a semiclassical string state with large R-charge and the light operator represented a BPS state – massless (supergravity) scalar or dilaton mode. A more refined approach based on vertex operator insertions was put forward in [12]. Further developments can be traced in subsequent papers [13]–[31]. At present the main efforts of the researchers worldwide are concentrated on the calculation of three-point functions of three heavy operators [32]–[38].

Parallel to these advances, an extension to four-point functions was initiated in [39, 40] and various correlation functions were computed. Furthermore, comparison with results from the gauge theory side was provided. Motivated by these studies, in the present paper we consider the subleading contributions to particular four-point correlation functions of two heavy BMN operators and two light chiral primary operators from the point of view of string theory in $AdS_5 \times S^5$. Thus we extend the findings of [39, 40]. We also examine particular four-point correlators of four heavy BMN operators, represented by an exchange diagram involving a chiral primary operator (CPO).

The paper is organized in the following way. To explain the method, in the next section we give a short review of the procedure for calculating semiclassically n -point correlation functions via vertex operators. Next, we proceed with the computation of subleading four-point correlators of two BMN operators and two CPOs. Furthermore, we investigate the leading contribution to four-point functions of four BMN operators. We conclude with a brief discussion on the results.

2 Correlation functions with two heavy operators

First we will review the case of two-point correlators. In the leading semiclassical approximation they are determined by the corresponding classical string solution [6]–[9]. If $V_{H1}(\xi_1)$ and $V_{H2}(\xi_2)$ are the two heavy vertex operators inserted at points ξ_1 and ξ_2 on the worldsheet, the two-point function for large string tension ($\sqrt{\lambda} \gg 1$) is given by the stationary point of the action

$$\langle V_{H1}(\xi_1) V_{H2}(\xi_2) \rangle \sim e^{-I}, \quad (2.1)$$

where I is the action of the $AdS_5 \times S^5$ superstring sigma model in the embedding coordinates

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\xi \left(\partial Y_M \bar{\partial} Y^M + \partial X_k \bar{\partial} X_k + \text{fermions} \right), \quad (2.2)$$

$$Y_M Y^M = -Y_0^2 - Y_5^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1, \quad X_k X_k = X_1^2 + \dots + X_6^2 = 1.$$

We work in conformal gauge and use a worldsheet with Euclidean signature. The 2D derivatives have the following form $\partial = \partial_1 + i\partial_2$, $\bar{\partial} = \partial_1 - i\partial_2$. The relation between the embedding coordinates, and the global and Poincaré coordinates of AdS_5 that we will need below is

$$Y_5 + iY_0 = \cosh \rho e^{it}, \quad Y_1 + iY_2 = \sinh \rho \cos \theta e^{i\phi_1}, \quad Y_3 + iY_4 = \sinh \rho \sin \theta e^{i\phi_2},$$

$$Y_m = \frac{x_m}{z}, \quad Y_4 = \frac{1}{2z}(-1 + z^2 + x^m x_m), \quad Y_5 = \frac{1}{2z}(1 + z^2 + x^m x_m), \quad (2.3)$$

where $x^m x_m = -x_0^2 + x_i x_i$ ($m = 0, 1, 2, 3$; $i = 1, 2, 3$). However, our approach necessitates the use of the Euclidean continuation of AdS_5

$$t_e = it, \quad Y_{0e} = iY_0, \quad x_{0e} = ix_0, \quad (2.4)$$

so that $Y_M Y^M = -Y_5^2 + Y_{0e}^2 + Y_i Y_i + Y_4^2 = -1$.

The stationary point solution solves the string equations of motion with singular sources provided by $V_{H1}(\xi_1)$ and $V_{H2}(\xi_2)$. Making use of the conformal symmetry of the theory, it is possible to map the ξ -plane to the Euclidean cylinder parameterized by (τ_e, σ)

$$e^{\tau_e + i\sigma} = \frac{\xi - \xi_2}{\xi - \xi_1}. \quad (2.5)$$

Under this conformal map the singular solution on the ξ -plane transforms into a smooth classical string solution on the cylinder [6, 7, 8]. The new solution has the same quantum numbers as the states represented by the vertex operators, so that no information is lost.

The discussion above can be repeated for a physical integrated vertex operator labeled by a point x on the boundary of the Poincaré patch of AdS_5 [4, 6]

$$V_H(x) = \int d^2\xi V_H(\xi; x), \quad V_H(\xi; x) \equiv V_H(z(\xi), x(\xi) - x, X_k(\xi)). \quad (2.6)$$

The two-point function $\langle V_{H1}(x_1) V_{H2}(x_2) \rangle$ is determined in a similar fashion by the classical action evaluated on the stationary point solution. After applying the conformal map to the cylinder (2.5) we obtain a smooth solution that is actually the corresponding solitonic string solution in terms of Poincaré coordinates which satisfies the boundary conditions [8]

$$\tau_e \rightarrow -\infty \implies z \rightarrow 0, \quad x \rightarrow x_1, \quad \tau_e \rightarrow +\infty \implies z \rightarrow 0, \quad x \rightarrow x_2. \quad (2.7)$$

As was shown in [12], the semiclassical three-point correlators with two heavy and one light operators are of the form

$$\begin{aligned} G_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \langle V_{H1}(\mathbf{x}_1) V_{H2}(\mathbf{x}_2) V_L(\mathbf{x}_3) \rangle \\ &= \int \mathcal{D}\mathbb{X}^{\mathbb{M}} e^{-I} \int d^2\xi_1 d^2\xi_2 d^2\xi_3 V_{H1}(\xi_1; \mathbf{x}_1) V_{H2}(\xi_2; \mathbf{x}_2) V_L(\xi_3; \mathbf{x}_3), \end{aligned} \quad (2.8)$$

where $\int \mathcal{D}\mathbb{X}^{\mathbb{M}}$ is the integral over (Y_M, X_k) . In the stationary point equations the contribution of the light operator can be neglected, so that the solution coincides with the one in the case of the two-point function of the two heavy operators. Thus we obtain [12]

$$\frac{G_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)}{G_2(\mathbf{x}_1, \mathbf{x}_2)} = \int d^2\xi V_L(z(\xi), x(\xi) - \mathbf{x}_3, X_k(\xi)), \quad (2.9)$$

where $(z(\xi), x(\xi), X_k(\xi))$ stands for the corresponding string solution with the same quantum numbers as the heavy vertex operators, transformed to the ξ -plane by (2.5). Taking account of the fact that $\int d^2\sigma = \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} d\sigma$, we can also represent (2.9) in terms of the cylinder

$$\frac{G_3}{G_2} = \int d^2\sigma V_L(z(\tau_e, \sigma), x(\tau_e, \sigma) - \mathbf{x}_3, X_k(\tau_e, \sigma)). \quad (2.10)$$

The global conformal $SO(2, 4)$ symmetry fixes the two- and three-point correlation functions (assuming that $V_{H2} = V_{H1}^*$)

$$G_2(\mathbf{x}_1, \mathbf{x}_2) = \frac{C_{12} \delta_{\Delta_1, \Delta_2}}{x_{12}^{\Delta_1 + \Delta_2}}, \quad x_{ij} \equiv |\mathbf{x}_i - \mathbf{x}_j|, \quad (2.11)$$

$$G_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{C_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{13}^{\Delta_1 + \Delta_3 - \Delta_2} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1}}, \quad (2.12)$$

where Δ_i are the scaling dimensions of the operators. With a proper choice of \mathbf{x}_i one can remove the dependence on x_{ij} in (2.10), and use (2.10) to compute the structure constants C_{123} [12]. Presuming that $\Delta_1 = \Delta_2$, we find after setting $C_{12} = 1$ in (2.11) that

$$\frac{G_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = 0)}{G_2(\mathbf{x}_1, \mathbf{x}_2)} = C_{123} \left(\frac{x_{12}}{|\mathbf{x}_1| |\mathbf{x}_2|} \right)^{\Delta_3}. \quad (2.13)$$

The leading contribution to the four-point functions of two heavy and two light operators is provided by

$$\begin{aligned} G_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \langle V_{H1}(\mathbf{x}_1) V_{H2}(\mathbf{x}_2) V_{L1}(\mathbf{x}_3) V_{L2}(\mathbf{x}_4) \rangle \\ &= \int \mathcal{D}\mathbb{X}^{\mathbb{M}} e^{-I} \int d^2\xi_1 d^2\xi_2 d^2\xi_3 d^2\xi_4 V_{H1}(\xi_1; \mathbf{x}_1) V_{H2}(\xi_2; \mathbf{x}_2) V_{L1}(\xi_3; \mathbf{x}_3) V_{L2}(\xi_4; \mathbf{x}_4). \end{aligned} \quad (2.14)$$

The semiclassical trajectory being the same, G_4 is determined by the product of the light operators on the solution

$$\frac{G_4(x_1, x_2, x_3, x_4)}{G_2(x_1, x_2)} = \int d^2\xi_3 V_{L1}(z(\xi_3), x(\xi_3) - x_3, X_k(\xi_3)) \int d^2\xi_4 V_{L2}(z(\xi_4), x(\xi_4) - x_4, X_k(\xi_4)). \quad (2.15)$$

Transforming to the (τ_e, σ) -coordinates we get

$$\frac{G_4}{G_2} = \int d^2\sigma d^2\sigma' V_{L1}(z(\tau_e, \sigma), x(\tau_e, \sigma) - x_3, X_k(\tau_e, \sigma)) V_{L2}(z(\tau'_e, \sigma'), x(\tau'_e, \sigma') - x_4, X_k(\tau'_e, \sigma')). \quad (2.16)$$

3 Subleading four-point functions at strong coupling

In this section we will consider the subleading contribution depicted on fig. 1 to the four-point correlator of two heavy BMN operators and two chiral primary operators (CPO) in $\mathcal{N} = 4$ SYM. The semiclassical string trajectory is determined by the string solution in [8] for generic positions of the heavy vertex operators. We will provide leading corrections to results in [39].

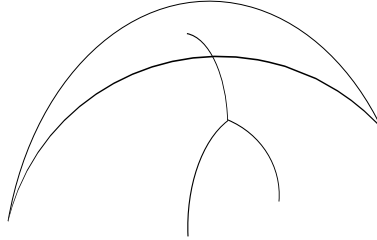


Figure 1: Subleading Witten diagram for the four-point function of two heavy and two light states.

The large-spin gauge theory operators in the correlation functions we will consider are dual to point-like strings with spin J in S^5 [8]. Nevertheless, here we start with generic positions $x_1 = -x_2$ of the heavy operator insertions. The corresponding Euclidean stationary point solution is [39]¹

$$z = \frac{x_{12}}{2\cosh(\kappa\tau_e)}, \quad x_{0e} = \frac{x_{12}}{2} \tanh(\kappa\tau_e), \quad x_i = 0, \quad \varphi = -i\kappa\tau_e, \quad \kappa = \frac{J}{\sqrt{\lambda}}, \quad (3.1)$$

where φ is a coordinate of S^5 . Ignoring for now any contribution from the light operators, we denote the respective heavy vertex operator as V_J , with $V_{-J} \equiv V_J^*$. The two-point function of such operators looks like [7, 8]

$$\langle V_{-J}(x_1) V_J(x_2) \rangle = \frac{1}{x_{12}^{\Delta_1 + \Delta_2}}, \quad \Delta_1 = \Delta_2 = J. \quad (3.2)$$

¹We choose the points x_1 and x_2 to lie on the x_{0e} -axis.

The massless string state corresponding to the CPO originates from the trace of the graviton in the S^5 directions [41, 42]. Building on results in [10, 43], we conjecture that the bosonic part of the respective light vertex operator has the following form²

$$V'_L(z, x) = V_{\Delta}^{(\text{CPO})}(z, x) = \hat{c}_{\Delta} A(z, x; \mathbf{x}_3, \mathbf{x}_4) e^{ij\varphi} [z^{-2}(\partial x_m \bar{\partial} x^m - \partial z \bar{\partial} z) - \partial X_k \bar{\partial} X_k]. \quad (3.3)$$

Here \mathbf{x}_3 and \mathbf{x}_4 ($\mathbf{x}_3 < \mathbf{x}_4$) are the positions of the CPOs with scaling dimensions Δ_3 and Δ_4 on the boundary, and [44]

$$A(z, x; \mathbf{x}_3, \mathbf{x}_4) = \int_H \frac{d^5 w}{w_0^5} G_{\Delta}(z, x; w) K_{\Delta_3}(w, \mathbf{x}_3) K_{\Delta_4}(w, \mathbf{x}_4), \quad (3.4)$$

where G_{Δ} denotes a CPO bulk-to-bulk propagator and K_{Δ} is a CPO bulk-to-boundary propagator. The normalization constant \hat{c}_{Δ} of the CPO is [10, 43]

$$\hat{c}_{\Delta} = \hat{c}_j = \frac{\sqrt{\lambda}}{8\pi N} (j+1) \sqrt{j}. \quad (3.5)$$

The light gauge theory operator is $\text{Tr} Z^j$ with conformal dimension $\Delta = j \geq 2$.

The subleading four-point functions differ significantly from the ones described in section 2. There four-point correlators were given in terms of products of three-point functions. In the present considerations the bulk-to-bulk propagator entering (3.3) through (3.4) introduces severe computational difficulties. More specifically, the four-point correlator assumes the explicit form [44]

$$\frac{G_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)}{G_2(\mathbf{x}_1, \mathbf{x}_2)} = \hat{c}_j \int d^2 \xi A(z, x; \mathbf{x}_3, \mathbf{x}_4) e^{ij\varphi} [z^{-2}(\partial x_m \bar{\partial} x^m - \partial z \bar{\partial} z) - \partial X_k \bar{\partial} X_k], \quad (3.6)$$

$$A(z, x; \mathbf{x}_3, \mathbf{x}_4) = \frac{\mathbf{x}_{34}^{-2\Delta_4} z^{\Delta_{34}}}{(z^2 + x^2)^{\Delta_{34}}} {}_2F_1\left(\frac{\Delta - \Delta_{34}}{2}, 2 - \frac{\Delta + \Delta_{34}}{2}; 2; 1 - \frac{1}{\zeta}\right), \quad \Delta_{34} = \Delta_3 - \Delta_4, \quad (3.7)$$

where

$$\zeta = \frac{\mathbf{x}_{34}^2 z^2}{(z^2 + x^2)[z^2 + (x + \mathbf{x}_{34})^2]}. \quad (3.8)$$

Evaluated on (3.1), (3.6) can be expressed as

$$\frac{G_4}{G_2} = \frac{4\pi 2^{\Delta_{34}} \hat{c}_j \kappa^2}{\mathbf{x}_{12}^{\Delta_{34}} \mathbf{x}_{34}^{2\Delta_4}} \int_{-\infty}^{\infty} d\tau_e \frac{{}_2F_1\left(\frac{\Delta - \Delta_{34}}{2}, 2 - \frac{\Delta + \Delta_{34}}{2}; 2; 1 - \frac{1}{\zeta(\kappa\tau_e)}\right) e^{j\kappa\tau_e}}{\cosh^{\Delta_{34}+2}(\kappa\tau_e)}. \quad (3.9)$$

In order to calculate the integral we consider extremal correlators. Thus we will get relations among Δ , Δ_3 and Δ_4 . First, let us concentrate on the simplest case $\Delta - \Delta_{34} = 0$. Then the integral simplifies enormously

$$\frac{G_4}{G_2} = \frac{4\pi 2^j \hat{c}_j \kappa^2}{\mathbf{x}_{12}^j \mathbf{x}_{34}^{2\Delta_4}} \int_{-\infty}^{\infty} d\tau_e \frac{e^{j\kappa\tau_e}}{\cosh^{j+2}(\kappa\tau_e)}. \quad (3.10)$$

²We ignore derivative terms that will not influence the calculation below since we work with $\mathbf{x}_1 = -\mathbf{x}_2$.

We change the variable $\kappa\tau_e \rightarrow \tau_e$, and obtain

$$\frac{G_4}{G_2} = \frac{8\pi 4^j \hat{c}_j \kappa}{(j+1)x_{12}^j x_{34}^{2\Delta_4}} = \frac{4^j J \sqrt{j}}{N x_{12}^j x_{34}^{2\Delta_4}}, \quad (3.11)$$

which leads to the following expression for the four-point function

$$G_4(x_1, x_2, x_3, x_4) = \frac{4^j J \sqrt{j}}{N x_{12}^{2J+j} x_{34}^{2\Delta_4}} = \frac{4^j J \sqrt{j}}{N x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{34}}{x_{12}} \right)^j, \quad (3.12)$$

which conforms to the required form of a four-point correlator. In the case of $j = 2$ we obtain

$$G_4(x_1, x_2, x_3, x_4) = \frac{16\sqrt{2}J}{N x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{34}}{x_{12}} \right)^2. \quad (3.13)$$

Now let us consider the channel $\Delta + \Delta_{34} = 0$. In this case the integral assumes the following form

$$\frac{G_4}{G_2} = \frac{4\pi \hat{c}_j \kappa^2}{x_{34}^{2\Delta_3}} \int_{-\infty}^{\infty} d\tau_e \frac{z^\Delta e^{j\kappa\tau_e}}{[z^2 + (x + x_{34})^2]^\Delta \cosh^2(\kappa\tau_e)}. \quad (3.14)$$

We change the variable $\kappa\tau_e \rightarrow \tau_e$, and obtain

$$\frac{G_4}{G_2} = \frac{4\pi 2^j \hat{c}_j \kappa}{x_{12}^{-j} x_{34}^{2\Delta_3}} \int_{-\infty}^{\infty} d\tau_e \frac{e^{j\tau_e}}{\cosh^{j+2}(\tau_e) [x_{12}^2 + 4x_{34}^2 + 4x_{12}x_{34} \tanh(\tau_e)]^j}. \quad (3.15)$$

Again we change the integration variable $\tau_e \rightarrow e^{2\tau_e} \equiv s$

$$\frac{G_4}{G_2} = \frac{8\pi \hat{c}_j \kappa (1-\eta)^j}{2^j x_{34}^{\Delta_3+\Delta_4}} \int_0^\infty ds \frac{s^j}{(1+s)^2 (\eta+s)^j}, \quad \eta \equiv \frac{(x_{12} - 2x_{34})^2}{(x_{12} + 2x_{34})^2}. \quad (3.16)$$

This integral can be calculated in a convenient way via differentiation by η^{-1} . The final result is

$$\frac{G_4}{G_2} = \frac{8\pi \hat{c}_j \kappa j}{2^j x_{34}^{\Delta_3+\Delta_4}} \left(\frac{\eta}{1-\eta} \sum_{k=1}^j \frac{(1-\eta)^k}{k} + \frac{(1-\eta)^j}{j} + \frac{\eta \ln \eta}{1-\eta} \right). \quad (3.17)$$

In other words

$$G_4 = \frac{J(j+1)j\sqrt{j}}{N 2^j x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{\eta}{1-\eta} \sum_{k=1}^j \frac{(1-\eta)^k}{k} + \frac{(1-\eta)^j}{j} + \frac{\eta \ln \eta}{1-\eta} \right). \quad (3.18)$$

This expression indeed bears all the expected characteristics of a conformal four-point function. The presence of only one cross-ratio is explained by the fact that the positions of boundary operators are heavily constrained. For the simplest case of $j = 2$ we get

$$G_4(x_1, x_2, x_3, x_4) = \frac{3\sqrt{2}J}{2N x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{1+\eta}{2} + \frac{\eta \ln \eta}{1-\eta} \right). \quad (3.19)$$

4 Four-point functions of heavy BMN states

In this section we will consider particular constrained four-point correlators of four heavy BMN operators corresponding to the string solution in [8] for generic positions of the heavy operator insertions. The correlation function can be thought of as two two-point correlators connected with a light bulk-to-bulk propagator (see fig. 2). We will follow the spirit of the calculations presented in [39].

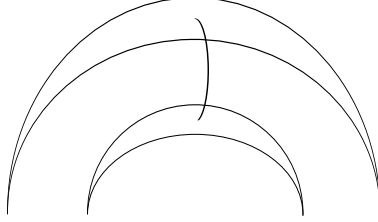


Figure 2: Witten diagram for the four-point function of four heavy BMN states.

The large-spin gauge theory operators in the correlator at hand are dual to the point-like in AdS strings with one angular momentum in S^5 [8]. The positions of the heavy operators on the boundary are chosen in the form $x_1 = -x_4$ and $x_2 = -x_3$. The corresponding Euclidean stationary point solutions for the two semiclassical propagators in fig. 2 are [39]³

$$z = \frac{x_{14}}{2\cosh(\kappa\tau_e)}, \quad x_{0e} = \frac{x_{14}}{2} \tanh(\kappa\tau_e), \quad x_i = 0, \quad \varphi_1 = -i\kappa\tau_e, \quad \kappa = \frac{J}{\sqrt{\lambda}}, \quad (4.1a)$$

$$z' = \frac{x_{23}}{2\cosh(\kappa'\tau'_e)}, \quad x'_{0e} = \frac{x_{23}}{2} \tanh(\kappa'\tau'_e), \quad x'_i = 0, \quad \varphi'_1 = -i\kappa'\tau'_e, \quad \kappa' = \frac{J'}{\sqrt{\lambda}}, \quad (4.1b)$$

and we assume without loss of generality that $x_{14} \geq x_{23}$. We denote the respective heavy vertex operators as V_J and $V_{J'}$, with $V_{-J} \equiv V_J^*$ and $V_{-J'} \equiv V_{J'}^*$. The two-point function of such operators can be calculated [7, 8]

$$\langle V_{-J}(x_1)V_J(x_4) \rangle = \frac{1}{x_{14}^{2\Delta(J)}}, \quad \langle V_{-J'}(x_2)V_{J'}(x_3) \rangle = \frac{1}{x_{23}^{2\Delta(J')}}, \quad \Delta(J) = J, \quad \Delta(J') = J'. \quad (4.2)$$

In order to obtain the four-point correlator we need to join two two-point functions with a light state. We choose the light operator to be the chiral primary operator. The bosonic part of the corresponding supergravity “vertex” operator is conjectured to have

³We choose the points x_i , $i = 1, \dots, 4$, to lie on the x_{0e} -axis.

the following form

$$\begin{aligned}
\mathbf{V}_L(z, x; z', x') &= \mathbf{V}_\Delta^{(\text{CPO})}(z, x; z', x') = \hat{c}_\Delta^2 G_\Delta(z, x; z', x') X^j X'^{-j} \mathcal{L} \mathcal{L}', \quad \Delta = j, \quad (4.3) \\
G_\Delta(z, z; z', x') &= \frac{2^{-\Delta}(\Delta-1)\zeta^\Delta}{2\pi^2} {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta-1; \zeta^2\right), \quad \zeta = \frac{2zz'}{z^2 + z'^2 + (x-x')^2}, \\
X &\equiv X_1 + iX_2 = \sin\gamma \cos\psi e^{i\varphi_1}, \quad X' \equiv X'_1 + iX'_2 = \sin\gamma' \cos\psi' e^{i\varphi'_1}, \\
\mathcal{L} &= \frac{\partial x_m \bar{\partial} x^m - \partial z \bar{\partial} z}{z^2} - \partial X_k \bar{\partial} X_k, \quad \mathcal{L}' = \frac{\partial' x'_m \bar{\partial}' x'^m - \partial' z' \bar{\partial}' z'}{z'^2} - \partial' X'_k \bar{\partial}' X'_k,
\end{aligned}$$

where the normalization constant \hat{c}_Δ of the CPO is given again by (3.5). The four-point correlator assumes the form

$$\frac{G_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)}{G_2(\mathbf{x}_1, \mathbf{x}_4)G_2'(\mathbf{x}_2, \mathbf{x}_3)} = \int d^2\xi d^2\xi' \mathbf{V}_\Delta^{(\text{CPO})}(z, x; z', x'). \quad (4.4)$$

Evaluated on (4.1), it can be presented as

$$\frac{G_4}{G_2 G_2'} = 16\pi^2 \hat{c}_j^2 \kappa^2 \kappa'^2 \int_{-\infty}^{\infty} d\tau_e \int_{-\infty}^{\infty} d\tau'_e \frac{G_\Delta e^{j(\kappa\tau_e - \kappa'\tau'_e)}}{\cosh^2(\kappa\tau_e) \cosh^2(\kappa'\tau'_e)}. \quad (4.5)$$

It can be shown that the bulk-to-bulk propagator for the CPO can be expressed in terms of elementary functions in the following way

$$G_\Delta = \frac{\zeta^j [(1-\zeta^2)^{-1/2} + j - 2]}{8\pi^2 (1-\zeta^2)(1+\sqrt{1-\zeta^2})^{j-2}}. \quad (4.6)$$

Let us change the integration variables from τ_e and τ'_e to $y \equiv \tanh(\kappa\tau_e)$ and $y' \equiv \tanh(\kappa'\tau'_e)$. Then (4.5) becomes

$$\frac{G_4}{G_2 G_2'} = 2\hat{c}_j^2 \kappa \kappa' \eta^j \int_{-1}^1 dy \int_{-1}^1 dy' \frac{(1+y)^j (1-y')^j}{(1-\eta y y')^j} \frac{(1-\zeta^2)^{-1/2} + j - 2}{(1-\zeta^2)(1+\sqrt{1-\zeta^2})^{j-2}}, \quad (4.7)$$

where we have introduced

$$\eta = \frac{2\mathbf{x}_{14}\mathbf{x}_{23}}{\mathbf{x}_{14}^2 + \mathbf{x}_{23}^2}.$$

We finally obtain for the four-point correlator

$$\begin{aligned}
\frac{G_4}{G_2 G_2'} &= 2\hat{c}_j^2 \kappa \kappa' \eta^j \int_{-1}^1 dy \int_{-1}^1 dy' [I_1 + (j-2)I_2], \quad (4.8) \\
I_1 &= \frac{(1+y)^j (1-y')^j (1-\eta y y')}{P^{3/2} (1-\eta y y' + \sqrt{P})^{j-2}}, \quad I_2 = \frac{(1+y)^j (1-y')^j}{P (1-\eta y y' + \sqrt{P})^{j-2}},
\end{aligned}$$

where $P = \eta^2(y^2 + y'^2) - 2\eta y y' + 1 - \eta^2$. If we choose for simplicity $j = 2$, we will get

$$G_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \frac{C_{J,J',j=2}}{\mathbf{x}_{14}^{2\Delta(J)} \mathbf{x}_{23}^{2\Delta(J')}} \quad (4.9)$$

where

$$C_{J,J',2} = 2\hat{c}_2^2 \kappa \kappa' \eta^2 \int_{-1}^1 dy \int_{-1}^1 dy' \frac{(1+y)^2(1-y')^2(1-\eta yy')}{P^{3/2}} \\ = \frac{9JJ'}{15\pi^2 N^2} \left(\frac{(2\chi-1)(3\chi-1)}{2\chi} - \frac{(3\chi-2)\ln\chi}{\chi-1} \right), \quad \chi = \frac{(x_{14} - x_{23})^2}{(x_{14} + x_{23})^2}. \quad (4.10)$$

5 Conclusion

The duality between gauge and string theories passed through many crucial developments over the last years. The AdS/CFT correspondence has achieved impressive results about the anomalous dimensions of gauge theory operators, integrable structures, etc., and various properties of gauge theories at strong coupling have been established. One of the main challenges ahead is to find efficient methods for calculation of correlation functions.

Although the three-point correlator of three heavy operators is not yet fully understood [32]–[37], we have discovered a lot about the behavior of correlation functions containing two heavy and one light states at strong coupling [10, 11].⁴ Using the trajectory for the correlator of two heavy operators [6, 8], the authors of [12] put forward an approach based on insertion of vertex operators. The same method applies also to higher n -point correlation functions with 2 heavy and $n - 2$ light operators. Namely, the semiclassical expression for the n -point correlator should be given by a product of light vertex operators evaluated on a worldsheet surface governed by the heavy operator insertions.

In the present paper we consider string theory on $AdS_5 \times S^5$ and compute subleading four-point functions at strong coupling, applying the ideas of [12] for calculation of correlators with vertex operators. We examine the method in the case of two BMN operators and two light CPOs. Moreover, we investigate specific four-point correlation functions of four string (BMN) states at strong coupling. We provide a number of limiting cases. Our considerations extend the results presented in [39, 40].

There has been also development in the computation of three- and four-point correlation functions with heavy operators via integrability techniques [45]–[51]. It would be very interesting to see if our results can be obtained in such a way from the gauge theory point of view.

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⁴These considerations were initiated in [9].

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